JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **192**, 543–561 (1995)

Pseudo Almost Periodic Solutions of Some Differential Equations, II

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Submitted by Hal L. Smith

Received May 12, 1993

We first present some results about pseudo almost periodic functions. Then we use these results to show some theorems about existence and uniqueness of the pseudo almost periodic solutions of the linear and the quasi-linear nonautonomous equations. Finally we set up some theorems of the existence of pseudo almost periodic solutions implying the existence of almost periodic solutions of general ordinary differential equation systems. © 1995 Academic Press, Inc.

In [6], we define and investigate the pseudo almost periodic functions, which are new generalizations of the almost periodic functions. In this paper we will continue our work in [7] to explore some applications of such functions in the theory of differential equations. For the pseudo almost periodic solutions of ordinary differential equations, we investigated some autonomous equations in Section 2 of [7]. In this paper, we will deal with some nonautonomous equations. To start, we recall and prove some results about the pseudo almost periodic functions in Section 1. Then we apply these results to investigate the existence and uniqueness of the pseudo almost periodic solutions of some linear and quasi-linear differential equations in Sections 2 and 3 respectively. Finally, in Section 4 we set up some theorems of the existence of the pseudo almost periodic solutions implying the existence of the almost periodic solutions of general ordinary differential equations.

0022-247X/95 $6.00

1. Pseudo Almost Periodic Functions

Let $\Omega \subset \mathbb{C}^n$ be closed and let $\mathscr{C}(\mathbb{R})$ (respectively, $\mathscr{C}(\Omega \times \mathbb{R})$ be the space of bounded continuous complex-valued functions on $\mathbb{R}$ (respectively, $\Omega \times \mathbb{R}$) with supremum norm. $|\cdot|$ denotes the usual Euclidean norm in $\mathbb{C}^n$. Also m denotes Lebesgue measure on $\mathbb{R}$.

We will call a function space C^*-algebra if it is a Banach space and is also closed under multiplication and complex conjugation. Both $\mathscr{C}(\mathbb{R})$ and $\mathscr{C}(\Omega \times \mathbb{R})$ are C^*-algebras.

A subspace $\mathscr{A}$ of $\mathscr{C}(\mathbb{R})$ ($\mathscr{C}(\Omega \times \mathbb{R})$) is known as an ideal if $f g \in \mathscr{A}$ whenever $f \in \mathscr{A}$ and $g \in \mathscr{C}(\mathbb{R})$ ($\mathscr{C}(\Omega \times \mathbb{R})$). A subspace $\mathscr{A}$ of $\mathscr{C}(\mathbb{R})$ is said to be translation invariant if the functions $f(\cdot + t)$ are in $\mathscr{A}$ for all functions $f \in \mathscr{A}$ and $t \in \mathbb{R}$.

Definition 1.1. A function $g \in \mathscr{C}(\mathbb{R})$ is called almost periodic if for each $\varepsilon > 0$, there exists an $l_\varepsilon > 0$ such that every interval of length l_ε contains a number τ with the property that

$$|g(t + \tau) - g(t)| < \varepsilon \qquad (t \in \mathbb{R}).$$

The number τ is called an ε-translation number of g. Denote by $\mathscr{AP}(\mathbb{R})$ the set of all such functions.

Definition 1.2. A function $g \in \mathscr{C}(\Omega \times \mathbb{R})$ is called almost periodic in $t \in \mathbb{R}$, uniformly with respect to $Z \in \Omega$, if for any $\varepsilon > 0$ corresponds a number $l_\varepsilon > 0$ such that any interval in $\mathbb{R}$ of length l_ε contains a number τ for which

$$|g(Z, t + \tau) - g(Z, t)| < \varepsilon \qquad (Z \in \Omega, t \in \mathbb{R}).$$

Denote by $\mathscr{AP}(\Omega \times \mathbb{R})$ the set of all such functions.

For the details in the theory of the amost periodic functions, we refer to [2] and its references.

Set

$$\mathscr{PAP}_0(\mathbb{R}) = \left\{\varphi \in \mathscr{C}(\mathbb{R}) : \lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} |\varphi(s)|\, ds = 0\right\}$$

and

$$\mathscr{PAP}_0(\Omega \times \mathbb{R}) = \left\{\varphi \in \mathscr{C}(\Omega \times \mathbb{R}) : \lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} |\varphi(Z, x)|\, dx \right.$$
$$\left. = 0 \text{ uniformly in } Z \in \Omega \right\}.$$

$\mathcal{PAP}_0(\mathbb{R})$ and $\mathcal{PAP}_0(\Omega \times \mathbb{R})$ are closed ideals of $\mathcal{C}(\mathbb{R})$ and $\mathcal{C}(\Omega \times \mathbb{R})$, respectively.

DEFINITION 1.3. A function $f \in \mathcal{C}(\mathbb{R})$ ($\mathcal{C}(\Omega \times \mathbb{R})$) is called pseudo almost periodic if

$$f = g + \varphi,$$

where $g \in \mathcal{AP}(\mathbb{R})$ ($\mathcal{AP}(\Omega \times \mathbb{R})$) and $\varphi \in \mathcal{PAP}_0(\mathbb{R})$ ($\mathcal{PAP}_0(\Omega \times \mathbb{R})$). The functions g and φ are called the almost periodic component and the ergodic perturbation respectively of the function f. Denote by $\mathcal{PAP}(\mathbb{R})$ ($\mathcal{PAP}(\Omega \times \mathbb{R})$) the set of all such functions f.

The following theorem is [7, Lemma 1.3].

THEOREM 1.4. *If $f \in \mathcal{PAP}(\mathbb{R})$ and if g is its almost periodic component, then we have*

$$g(\mathbb{R}) \subset \overline{f(\mathbb{R})}.$$

Therefore $\|f\| \geq \|g\| \geq \inf_{x\in\mathbb{R}} |g(t)| \geq \inf_{t\in\mathbb{R}} |f(t)|$.

THEOREM 1.5. *$\mathcal{PAP}(\mathbb{R})$ is a translation invariant C^*-subalgebra of $\mathcal{C}(\mathbb{R})$ containing the constant functions. Furthermore,*

$$\mathcal{PAP}(\mathbb{R}) = \mathcal{AP}(\mathbb{R}) + \mathcal{PAP}_0(\mathbb{R});$$

that is, the function $f \in \mathcal{PAP}(\mathbb{R})$ has a unique decomposition.

Proof. The first statement is [7, Theorem 1.4]. Now we use the previous theorem to show the second statement. Let $f \in \mathcal{PAP}(\mathbb{R})$. Suppose there are $g_i \in \mathcal{AP}(\mathbb{R})$ and $\varphi_i \in \mathcal{PAP}_0(\mathbb{R})$, $i = 1, 2$, such that $f = g_i + \varphi_i$. Then $0 = (g_1 - g_2) + (\varphi_1 - \varphi_2)$. Since both $\mathcal{AP}(\mathbb{R})$ and $\mathcal{PAP}_0(\mathbb{R})$ are linear, we have that $g_1 - g_2 \in \mathcal{AP}(\mathbb{R})$ and $\varphi_1 - \varphi_2 \in \mathcal{PAP}_0(\mathbb{R})$. By Theorem 1.4, $(g_1 - g_2)(\mathbb{R}) \subset \{0\}$. Thus, $g_1 = g_2$ and, therefore, $\varphi_1 = \varphi_2$.

DEFINITION 1.6. A closed subset C of $\mathbb{R}$ is said to be an ergodic zero set in $\mathbb{R}$ if $m(C \cap [-t, t])/2t \to 0$ as $t \to \infty$.

The proof of the following theorem is straightforward.

THEOREM 1.7. *A function $\varphi \in \mathcal{C}(\mathbb{R})$ is in $\mathcal{PAP}_0(\mathbb{R})$ if and only if, for $\varepsilon > 0$, the set $C_\varepsilon = \{t \in \mathbb{R} : |\varphi(t)| \geq \varepsilon\}$ is an ergodic zero subset in $\mathbb{R}$.*

THEOREM 1.8. *The following statements hold:*

(1) *A function $\varphi \in \mathcal{C}(\mathbb{R})$ is in $\mathcal{PAP}_0(\mathbb{R})$ if and only if φ^2 is.*

(2) *$\Phi \in \mathcal{C}(\mathbb{R})^n$ is in $\mathcal{PAP}_0(\mathbb{R})^n$ if and only if the norm function $|\Phi(\cdot)|$ is in $\mathcal{PAP}_0(\mathbb{R})$.*

Proof. (1) The sufficiency follows since

$$\frac{1}{2t}\int_{-t}^{t}|\varphi(x)|\,dx \le \frac{1}{2t}\left[\int_{-t}^{t}|\varphi(x)|^2\,dx\right]^{1/2}\left[\int_{-t}^{t}1\,dx\right]^{1/2} = \left[\frac{1}{2t}\int_{-t}^{t}|\varphi(x)|^2\,dx\right]^{1/2}.$$

The necessity follows from the fact that $\mathcal{PAP}_0(\mathbb{R})$ is an ideal of $\mathcal{C}(\mathbb{R})$.

(2) By (1), $\Phi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \in \mathcal{PAP}_0(\mathbb{R})^n$ if and only if $\varphi_i\overline{\varphi_i} \in \mathcal{PAP}_0(\mathbb{R})$, $i = 1, 2, \ldots, n$. The latter is equivalent to $|\Phi(\cdot)|^2 = \sum_{i=1}^{n} |\varphi_i(\cdot)|^2 \in \mathcal{PAP}_0(\mathbb{R})$, which, again by (1), is equivalent to $|\Phi(\cdot)| \in \mathcal{PAP}_0(\mathbb{R})$.

For $H = (h_1, h_2, \ldots, h_n) \in \mathcal{C}(\mathbb{R})^n$, suppose that $H(t) \in \Omega$ for all $t \in \mathbb{R}$. Define $H \times \iota\colon \mathbb{R} \to \Omega \times \mathbb{R}$ by

$$H \times \iota(t) = (h_1(t), h_2(t), \ldots, h_n(t), t) \qquad (t \in \mathbb{R}).$$

For $F = (f_1, f_2, \ldots, f_n) \in \mathcal{PAP}(\mathbb{R})^n$, let $G = (g_1, g_2, \ldots, g_n)$ and $\Phi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$, where g_i and φ_i are the almost periodic component and the ergodic perturbation, respectively, of f_i, $i = 1, 2, \ldots, n$.

Let K be a compact subset of Ω. A function $f \in \mathcal{C}(\Omega \times \mathbb{R})$ is said to be continuous in $Z \in K$ uniformly in $t \in \mathbb{R}$ if, for given $Z \in K$ and $\varepsilon > 0$, there exists a $\delta(Z, \varepsilon) > 0$ such that $Z' \in K$ and $|Z' - Z| < \delta(Z, \varepsilon)$ imply that $|f(Z', t) - f(Z, t)| < \varepsilon$ for all $t \in \mathbb{R}$.

It is obvious that if f satisfies a Lipschitz condition; that is, there is an $L > 0$ such that

$$|f(Z', t) - f(Z'', t)| \le L|Z' - Z''| \qquad (Z', Z'' \in K; t \in \mathbb{R}),$$

then f is continuous in $Z \in K$ uniformly in $t \in \mathbb{R}$. In this connection, the following theorem improves [7, Theorem 1.5].

THEOREM 1.9. *Suppose the function $f \in \mathcal{PAP}(\Omega \times \mathbb{R})$ is continuous in $Z \in K$ uniformly in $t \in \mathbb{R}$ for all compact subsets $K \subset \Omega$ and $F \in \mathcal{PAP}(\mathbb{R})^n$ such that $F(\mathbb{R}) \subset \Omega$, then $f \circ (F \times \iota) \in \mathcal{PAP}(\mathbb{R})$.*

Proof. Let $f = g + \varphi$ and $F = G + \Phi$ with $G = (g_1, g_2, \ldots, g_n) \in \mathcal{AP}(\mathbb{R})^n$, as above. Note that

$$\begin{aligned} f \circ (F \times \iota) &= g \circ (F \times \iota) + \varphi \circ (F \times \iota) \\ &= g \circ (G \times \iota) + [g \circ (F \times \iota) - g \circ (G \times \iota) + \varphi \circ (F \times \iota)]. \end{aligned}$$

It follows from Theorem 1.4 that $G(\mathbb{R}) \subset \overline{F(\mathbb{R})} \subset \Omega$. Therefore, $g \circ (G \times \iota) \in \mathcal{AP}(\mathbb{R})$ [2, Theorem 2.8]. To finish the proof, we need to show that the function $h = g \circ (F \times \iota) - g \circ (G \times \iota) + \varphi \circ (F \times \iota)$ is in $\mathcal{PAP}_0(\mathbb{R})$.

First we show that $g \circ (F \times \iota) - g \circ (G \times \iota) \in \mathcal{PAP}_0(\mathbb{R})$.

It is trivial in the case that $g = 0$. So we assume that $g \neq 0$. Set $\Omega_1 = \overline{F(\mathbb{R})}$. Then the function g is uniformly continuous on $\Omega_1 \times \mathbb{R}$ [2, Theorem 2.3]. For $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|g(Z_1, t) - g(Z_2, t)| < \frac{\varepsilon}{2} \qquad (Z_1, Z_2 \in \Omega_1, |Z_1 - Z_2| < \delta; t \in \mathbb{R}). \quad (1.1)$$

Set

$$C_\delta = \{t \in \mathbb{R} : |F(t) - G(t)| = |\Phi(t)| \geq \delta\}. \quad (1.2)$$

It follows from Theorems 1.7 and 1.8(2) that C_δ is an ergodic zero subset of $\mathbb{R}$. We can find $T > 0$ such that when $t \geq T$

$$\frac{m([-t, t] \cap C_\delta)}{2t} < \frac{\varepsilon}{4\|g\|}. \quad (1.3)$$

It follows from (1.1), (1.2), and (1.3) that

$$\begin{aligned} &\frac{1}{2t}\int_{-t}^{t} |g(F(s), s) - g(G(s), s)|\, ds \\ &= \frac{1}{2t}\left\{\int_{[-t,t]\setminus C_\delta} + \int_{[-t,t]\cap C_\delta} |g(F(s), s) - g(G(s), s)|\, ds\right\} \\ &\leq \frac{\varepsilon}{2} + 2\|g\| \frac{m([-t, t] \cap C_\delta)}{2t} < \varepsilon. \end{aligned}$$

Therefore, $g \circ (F \times \iota) - g \circ (G \times \iota) \in \mathcal{PAP}_0(\mathbb{R})$.

Finally, we show that $\varphi \circ (F \times \iota) \in \mathcal{PAP}_0(\mathbb{R})$. Note that $f = g + \varphi$ and g is uniformly continuous on $\Omega_1 \times \mathbb{R}$. By the hypothesis, f is continuous in $Z \in \Omega_1$ uniformly in $t \in \mathbb{R}$; so is φ. Since Ω_1 is compact in $\mathbb{C}^n$, one can find finite, say m, open balls O_k with center $Z^{(k)} \in \Omega_1$, $k = 1, 2, \ldots, m$, and radius $\delta(Z^{(k)}, \varepsilon/2)$ such that $\Omega_1 \subset \bigcup_{k=1}^{m} O_k$ and

$$|\varphi(Z, x) - \varphi(Z^{(k)}, x)| < \frac{\varepsilon}{2} \qquad (Z \in O_k, x \in \mathbb{R}). \quad (1.4)$$

The set

$$B_k = \{x \in \mathbb{R} : F(x) \in O_k\} \tag{1.5}$$

is open and $\mathbb{R} = \bigcup_{k=1}^{m} B_k$. Let $E_k = B_k \setminus \bigcup_{j=1}^{k-1} B_j$, then $E_k \cap E_j = \varnothing$ when $k \neq j$, $1 \leq k, j \leq m$.

Since each $\varphi(Z^{(k)}, \cdot) \in \mathcal{PAP}_0(\mathbb{R})$, there is a number $t_0 > 0$ such that

$$\sum_{k=1}^{m} \frac{1}{2t} \int_{-t}^{t} |\varphi(Z^{(k)}, x)|\, dx < \frac{\varepsilon}{2} \qquad (t \geq t_0). \tag{1.6}$$

It follows from (1.4), (1.5), and (1.6) that

$$\begin{aligned}
&\frac{1}{2t} \int_{-t}^{t} |\varphi(F(x), x)|\, dx \\
&\quad \leq \frac{1}{2t} \sum_{k=1}^{m} \int_{E_k \cap [-t,t]} |\varphi(F(x), x) - \varphi(Z^{(k)}, x)| + |\varphi(Z^{(k)}, x)|\, dx \\
&\quad \leq \frac{\varepsilon}{2} + \sum_{k=1}^{m} \frac{1}{2t} \int_{-t}^{t} |\varphi(Z^{(k)}, x)|\, dx \\
&\quad < \varepsilon.
\end{aligned}$$

This shows that $\varphi \circ (F \times \iota) \in \mathcal{PAP}_0(\mathbb{R})$. The proof is complete.

Define

$$\mathcal{C}_0(\Omega \times \mathbb{R}) = \{f \in \mathcal{C}(\Omega \times \mathbb{R}) : f(Z, t) \to 0, \text{ uniformly in } Z \in \Omega, \text{ as } |t| \to \infty\}.$$

Let $\mathcal{AAP}(\Omega \times \mathbb{R})$ denote all the functions f of the form

$$f = g + \varphi,$$

where $g \in \mathcal{AP}(\Omega \times \mathbb{R})$ and $\varphi \in \mathcal{C}_0(\Omega \times \mathbb{R})$. The members of $\mathcal{AAP}(\Omega \times \mathbb{R})$ are asymptotically almost periodic functions.

It is obvious that $\mathcal{C}_0(\Omega \times \mathbb{R}) \subset \mathcal{PAP}_0(\Omega \times \mathbb{R})$ and $\mathcal{AAP}(\Omega \times \mathbb{R}) \subset \mathcal{PAP}(\Omega \times \mathbb{R})$.

COROLLARY 1.10. *If $f \in \mathcal{AAP}(\Omega \times \mathbb{R})$ and $F \in \mathcal{AAP}(\mathbb{R})^n$ such that $F(\mathbb{R}) \subset \Omega$, then $f \circ (F \times \iota) \in \mathcal{AAP}(\mathbb{R})$.*

Proof. As in the proof of the previous theorem,

$$\begin{aligned}
f \circ (F \times \iota) &= g \circ (F \times \iota) + \varphi \circ (F \times \iota) \\
&= g \circ (G \times \iota) + [g \circ (F \times \iota) - g \circ (G \times \iota) + \varphi \circ (F \times \iota)],
\end{aligned}$$

where $g \circ (G \times \iota) \in \mathcal{AP}(\mathbb{R})$. By the hypothesis that $\Phi = F - G \in \mathscr{C}_0(\mathbb{R})^n$ and $\varphi \in \mathscr{C}_0(\Omega \times \mathbb{R})$, it follows that $g \circ (F \times \iota) - g \circ (G \times \iota) \in \mathscr{C}_0(\mathbb{R})$ because the uniform continuity of g and $\varphi \circ (F \times \iota) \in \mathscr{C}_0(\mathbb{R})$ because

$$\varphi(F(t), t) \leq \sup_{Z \in \Omega} \varphi(Z, t).$$

The proof is complete.

2. Linear Ordinary Differential Equations

Consider the nonautonomous equation system

$$\frac{dY}{dt} = A(t)Y + F \tag{2.1}$$

and its homogeneous system

$$\frac{dY}{dt} = A(t)Y, \tag{2.2}$$

where the $n \times n$ coefficient matrix $A(t)$ is continuous on $\mathbb{R}$ and column vector $F = (f_1, f_2, \ldots, f_n)'$ is in $\mathscr{C}(\mathbb{R})^n$. Define $\|F\| = \sup_{t \in \mathbb{R}} |F(t)|$. We will call $A(t)$ almost periodic if all the entries are almost periodic.

Definition 2.1. Equation (2.2) is said to satisfy an exponentional dichotomy if there exist a projection P on $\mathbb{C}^n$ and positive constants σ_i, k_i, $i = 1, 2$, such that

$$\begin{aligned} |X(t)PX^{-1}(s)| &\leq k_1 e^{-\sigma_1(t-s)} \qquad (t \geq s) \\ |X(t)(I - P)X^{-1}(s)| &\leq k_2 e^{-\sigma_2(s-t)} \qquad (t \leq s), \end{aligned} \tag{2.3}$$

for a fundamental solution matrix X, where I is the identity matrix.

By a projection,we mean a matrix P such that $P^2 = P$. The projection P in Definition 2.1 is indeed uniquely determined. In fact, if $X(0) = I$, let

$$\begin{aligned} Y_+ &= \{y : y \text{ is solution of (2.2) bounded on } [0, \infty)\}, \\ Y_- &= \{y : y \text{ is solution of (2.2) bounded on } (\infty, 0]\}, \\ E_+ &= \{y(0) \in \mathbb{C}^n : y \in Y_+\}, \\ E_- &= \{y(0) \in \mathbb{C}^n : y \in Y_-\}. \end{aligned}$$

Then the range of P is E_+ and the nullspace of P is E_-. See [1, Section 2] for details of this.

THEOREM 2.2. *Suppose the system (2.2) satisfies an exponential dichotomy and the function F in (2.1) is in $\mathcal{PAP}_0(\mathbb{R})^n$. Then the equation system (2.1) has a unique bounded solution Y and $Y \in \mathcal{PAP}_0(\mathbb{R})^n$.*

Proof. By checking directly, one shows that the function

$$Y(t) = \int_{-\infty}^{t} X(t)PX^{-1}(s)F(s)\,ds - \int_{t}^{\infty} X(t)(I-P)X^{-1}(s)F(s)\,ds \tag{2.4}$$

is a solution of (2.1). Now, we show that the solution is bounded. It follows from (2.4) that

$$\begin{aligned}
|Y(t)| &\le \int_{-\infty}^{t} \left\|X(t)PX^{-1}(s)\right\| |F(s)|\,ds + \int_{t}^{\infty} \left\|X(t)(I-P)X^{-1}(s)\right\| |F(s)|\,ds \\
&\le \|F\| \left[\int_{-\infty}^{t} k_1 e^{-\sigma_1(t-s)}\,ds + \int_{t}^{\infty} k_2 e^{-\sigma_2(s-t)}\,ds \right] \\
&= \|F\| \left[k_1 e^{-\sigma_1 t} \int_{-\infty}^{t} e^{\sigma_1 s}\,ds + k_2 e^{\sigma_2 t} \int_{t}^{\infty} e^{-\sigma_2 s}\,ds \right] \\
&= \|F\| \left[\frac{k_1}{\sigma_1} + \frac{k_2}{\sigma_2} \right].
\end{aligned}$$

The solution Y is bounded because F is bounded. The bounded solution is unique because the homogeneous equation (2.2) has no nontrivial bounded solution.

Next, we want to show that $Y \in \mathcal{PAP}_0(\mathbb{R})^n$. Let $I(t) = \int_{-\infty}^{t} X(t)PX^{-1}(s)F(s)\,ds$ and $II(t) = \int_{t}^{\infty} X(t)(I-P)X^{-1}(s)F(s)\,ds$. Then $Y = I + II$. It follows from (2.3) that

$$\begin{aligned}
&\frac{1}{2T}\int_{-T}^{T} |I(t)|\,dt \\
&\quad \le \frac{1}{2T}\int_{-T}^{T} dt \int_{-\infty}^{t} |X(t)PX^{-1}(s)|\,|F(s)|\,ds \\
&\quad \le \frac{1}{2T}\int_{-T}^{T} dt \int_{-\infty}^{t} k_1 e^{-\sigma_1(t-s)}|F(s)|\,ds \\
&\quad = \frac{1}{2T}\int_{-T}^{T} dt \int_{-\infty}^{-T} + \int_{-T}^{t} k_1 e^{-\sigma_1(t-s)}|F(s)|\,ds \\
&\quad = \frac{1}{2T}\int_{-\infty}^{-T} |F(s)|\,ds \int_{-T}^{T} k_1 e^{-\sigma_1(t-s)}\,dt + \frac{1}{2}\int_{-T}^{T} |F(s)|\,ds \int_{s}^{T} k_1 e^{-\sigma_1(t-s)}\,dt \\
&\quad = I_1 + I_2.
\end{aligned}$$

To show that $I \in \mathcal{PAP}_0(\mathbb{R})^n$, we need to show that both $I_1 \to 0$ and $I_2 \to 0$ when $T \to \infty$:

$$\begin{aligned} I_1 &= \frac{1}{2T}\int_{-\infty}^{-T} |F(s)|\, ds \int_{-T}^{T} k_1 e^{-\sigma_1(t-s)}\, dt \\ &\leq \frac{1}{2T}\|F\|\frac{k_1}{\sigma_1}[e^{\sigma_1 T} - e^{-\sigma_1 T}]\int_{-\infty}^{-T} e^{\sigma_1 s}\, ds \\ &= \frac{1}{2T}\|F\|\frac{k_1}{\sigma_1^2}[1 - e^{-2_1 T}], \end{aligned}$$

so $I_1 \to 0$ as $T \to \infty$;

$$\begin{aligned} I_2 &= \frac{1}{2T}\int_{-T}^{T} |F(s)|\, ds \int_{s}^{T} k_1 e^{-\sigma_1(t-s)}\, dt \\ &= \frac{1}{2T}\int_{-T}^{T} \frac{k_1}{\sigma_1}(1 - e^{\sigma_1(s-T)})|F(s)|\, ds \\ &\leq \frac{k_1}{\sigma_1}\frac{1}{2T}\int_{-T}^{T} |F(s)|\, ds, \end{aligned}$$

therefore $I_2 \to 0$ as $T \to \infty$ because $|F(\cdot)| \in \mathcal{PAP}_0(\mathbb{R})$ (Theorem 1.8 (2)).
Similarly one shows that $II \in \mathcal{PAP}_0(\mathbb{R})^n$. The proof is complete.

THEOREM 2.3. *Suppose $A(t)$ in* (2.2) *is almost periodic and* (2.2) *satisfies an exponential dichotomy. Then, for every $F \in \mathcal{PAP}(\mathbb{R})^n$, there corresponds a unique bounded solution $Y_F \in \mathcal{PAP}(\mathbb{R})^n$ in the equation system* (2.1). *The mapping $F \to Y_F$ is bounded linear with*

$$\|Y_F\| \leq (k_1/\sigma_1 + k_2/\sigma_2)\|F\|. \tag{2.5}$$

Proof. Since $F \in \mathcal{PAP}(\mathbb{R})^n$, $F = G + \Phi$, where $G \in \mathcal{AP}(\mathbb{R})^n$, and $\Phi \in \mathcal{PAP}_0(\mathbb{R})^n$. As in the proof of the previous theorem, the function

$$\begin{aligned} Y_F(t) &= \int_{-\infty}^{t} X(t)PX^{-1}(s)F(s)\, ds - \int_{t}^{\infty} X(t)(I-P)X^{-1}(s)F(s)\, ds \\ &= \left[\int_{-\infty}^{t} X(t)PX^{-1}(s)G(s)\, ds - \int_{t}^{\infty} X(t)(I-P)X^{-1}(s)G(s)\, ds\right] \qquad (2.6) \\ &\quad + \left[\int_{-\infty}^{t} X(t)PX^{-1}(s)\Phi(s)\, ds - \int_{t}^{\infty} X(t)(I-P)X^{-1}(s)\Phi(s)\, ds\right] \\ &= Y_G + Y_\Phi \end{aligned}$$

is the unique solution of (2.1). By [3, Theorem 7.7], $Y_G \in \mathcal{AP}(\mathbb{R})^n$. By Theorem 2.2, $Y_\Phi \in \mathcal{PAP}_0(\mathbb{R})^n$. Therefore, $Y_F \in \mathcal{PAP}(\mathbb{R})^n$. The mapping $F \to Y_F$ is obviously linear. The norm estimate (2.5) comes from the first equality of (2.6) and the exponential dichotomy.

3. Quasi-linear Ordinary Differential Equations

In this section, we first consider the following quasi-linear equation system

$$\frac{dY}{dt} = A(t)Y + F + \mu G \circ (Y \times \iota), \tag{3.1}$$

where $\mu \in \mathbb{C}\backslash\{0\}$, $A(t)$ is a $n \times n$ almost periodic matrix, $F \in \mathcal{PAP}(\mathbb{R})^n$, and $G \in \mathcal{PAP}(\Omega \times \mathbb{R})^n$. We call the equation system

$$\frac{dY}{dt} = A(t)Y + F \tag{3.2}$$

the generating system of (3.1).

By Theorem 2.3, the system (3.2) has a unique solution $Y^{(0)} \in \mathcal{PAP}(\mathbb{R})^n$ if (2.2) satisfies an exponential dichotomy. Now, we have the following theorem about (3.1).

THEOREM 3.1. *Let F and $A(t)$ be as above and (2.2) satisfies an exponential dichotomy. Let $Y^{(0)}$ be the unique solution in $\mathcal{PAP}(\mathbb{R})^n$ of the generating system of* (3.2), *let $a > 0$, and let $\Omega = \cup\{Z \in \mathbb{C}^n : |Z - Y^{(0)}(t)| \le a,\ t \in \mathbb{R}\}$. Assume that*

(1) *$G \in \mathcal{PAP}(\Omega \times \mathbb{R})^n$ such that*

$$|G(Z', t) - G(Z'', t)| \le L|Z' - Z''| \qquad (Z', Z'' \in \Omega, t \in \mathbb{R}), \tag{3.3}$$

where $L > 0$;

(2) *$0 < |\mu| < \min\{1/(k_1/\sigma_1 + k_2/\sigma_2)L,\ a/(k_1/\sigma_1 + k_2/\sigma_2)\|G\|\}$, where k_i and σ_i, $i = 1, 2$, are as in Theorem* 2.3.

Then there exists a unique solution $Y \in \mathcal{PAP}(\mathbb{R})^n$ of the system (3.1) *such that $Y(t) \in \Omega$ for all $t \in \mathbb{R}$. Furthermore, $\|Y - Y^{(0)}\| \to 0$ as $\mu \to 0$.*

Proof. We construct a sequence of approximations by induction, starting with $Y^{(0)}$, and taking as $Y^{(k)}$ the bounded solution of the system

$$\frac{dY^{(k)}}{dt} = A(t)Y^{(k)} + F + \mu G \circ (Y^{(k-1)} \times \iota). \tag{3.4}$$

First, we show that $Y^{(k)}$ exists, $Y^{(k)} \in \mathcal{PAP}(\mathbb{R})^n$, and $Y^{(k)}(\mathbb{R}) \subset \Omega$, $k = 0, 1, 2, \ldots$. The conclusion holds for $k = 0$ by the hypothesis. Let us assume that the conclusion holds for $k - 1$. Then we show the conclusion for k. Since $G \circ (Y^{(k-1)} \times \iota) \in \mathcal{PAP}(\mathbb{R})^n$ (Theorem 1.9), there is a unique solution $Y^{(k)} \in \mathcal{PAP}(\mathbb{R})^n$ of (3.4) (Theorem 2.3). It follows from (3.2) and (3.4) that

$$\frac{d[Y^{(k)} - Y^{(0)}]}{dt} = A(t)[Y^{(k)} - Y^{(0)}] + \mu G \circ (Y^{(k-1)} \times \iota).$$

It follows from (2.5) that

$$\|Y^{(k)} - Y^{(0)}\| \leq (k_1/\sigma_1 + k_2/\sigma_2)|\mu|\,\|G\|.$$

Therefore, $Y^{(k)}(\mathbb{R}) \subset \Omega$, since

$$|\mu| \leq a/(k_1/\sigma_1 + k_2/\sigma_2)\|G\|.$$

Next, we show that $\{Y^{(k)}\}$ is Cauchy in $\mathcal{PAP}(\mathbb{R})^n$. Since

$$\frac{d[Y^{(k+1)} - Y^{(k)}]}{dt} = A(t)[Y^{(k+1)} - Y^{(k)}] + \mu[G \circ (Y^{(k)} \times \iota) - G \circ (Y^{(k-1)} \times \iota)],$$

it follows from (2.5) and (3.3) that

$$\begin{aligned}
\|Y^{(k+1)} - Y^{(k)}\| &\leq (k_1/\sigma_1 + k_2/\sigma_2)|\mu|\,\|G \circ (Y^{(k)} \times \iota) - G \circ (Y^{(k-1)} \times \iota)\| \\
&\leq (k_1/\sigma_1 + k_2/\sigma_2)|\mu| L \|Y^{(k)} - Y^{(k-1)}\| \\
&= \alpha\|Y^{(k)} - Y^{(k-1)}\| \\
&\ \ \vdots \\
&\leq \alpha^k \|Y^{(1)} - Y^{(0)}\|,
\end{aligned}$$

where $0 < \alpha = (k_1/\sigma_1 + k_2/\sigma_2)|\mu|L < 1$. This shows that $\{Y^{(k)}\}$ is Cauchy in $\mathcal{PAP}(\mathbb{R})^n$. Since $\mathcal{PAP}(\mathbb{R})$ is a Banach space (Theorem 1.5), so is $\mathcal{PAP}(\mathbb{R})^n$. There is a $Y \in \mathcal{PAP}(\mathbb{R})^n$ such that $\|Y^{(k)} - Y\| \to 0$ when $k \to \infty$. It follows from (3.4) that Y is a solution of (3.1). It is clear that $\|Y - Y^{(0)}\| \to 0$ as $\mu \to 0$.

To show the uniqueness, let Y' be another solution of (3.1). Similar to the discussion above, we have

$$\|Y - Y'\| \leq \alpha \|Y - Y'\|,$$

a contradiction. The proof is complete.

For $r > 0$, let $\Omega_1 = \{z \in \mathbb{C} : |z| \leq r\}$. Now, consider a system of the form

$$\frac{dY}{dt} = A(t)Y + F + \mu G \circ (Y \times \iota, \mu), \tag{3.5}$$

where $A(t)$ and F are as above, G is bounded on $\Omega \times \mathbb{R} \times \Omega_1$, and, for fixed $\mu \in \Omega_1$, $G \in \mathcal{PAP}(\Omega \times \mathbb{R})^n$.

Similarly, we get the generating system of (3.5) by putting $\mu = 0$.

THEOREM 3.2. *Let the system* (2.2), $A(t)$, F, $Y^{(0)}$, a, *and* Ω *be as in Theorem* 3.1 *and let* Ω_1 *be as above. Assume that, for fixed* $\mu \in \Omega_1$, *the function* $G \in \mathcal{PAP}(\Omega \times \mathbb{R})^n$ *and is such that*

$$|G(Z', t, \mu) - G(Z'', t, \mu)| < L|Z' - Z''|,$$

where $L > 0$ *and* (Z', t, μ) $(Z'', t, \mu) \in \Omega \times \mathbb{R} \times \Omega_1$. *Then there exists* $\mu_0 > 0$ *such that* (3.5) *has a unique solution* $Y(\cdot, \mu)$ *for all* $\mu \in B = \{z \in \Omega_1 : |z| \leq \mu_0\}$ *and* $Y(\cdot, \mu) \in \mathcal{PAP}(\mathbb{R})^n$. *If* G *is uniformly continuous on* $\Omega \times \mathbb{R} \times \Omega_1$, *then* $Y \in \mathcal{PAP}(\mathbb{R} \times B)^n$.

Proof. Let $\alpha = (k_1/\sigma_1 + k_2/\sigma_2)\mu$. Note that G is bounded on $\Omega \times \mathbb{R} \times \Omega_1$. As in the proof of the previous theorem, there is a $\mu_0 > 0$ such that (3.5) has a unique pseudo almost periodic solution $Y(\cdot, \mu)$ for $\mu \in B$ with $\alpha L < 1$. We want to show that $Y \in \mathcal{PAP}(\mathbb{R} \times B)^n$ if G is uniformly continuous on $\Omega \times \mathbb{R} \times \Omega_1$.

First we show that Y is uniformly continuous on $\mathbb{R} \times B$. It follows from (2.5) that

$$\begin{aligned}
&\|Y(\cdot + \Delta t, \mu + \Delta\mu) - Y(\cdot, \mu)\| \\
&\leq \alpha[\|G(Y(\cdot + \Delta t, \mu + \Delta\mu), \cdot + \Delta t, \mu + \Delta\mu) - G(Y(\cdot, \mu), \cdot, \mu)\|] \\
&\leq \alpha[\|G(Y(\cdot + \Delta t, \mu + \Delta\mu), \cdot + \Delta t, \mu + \Delta\mu) - G(Y(\cdot, \mu), \cdot + \Delta t, \mu + \Delta\mu)\| \\
&\quad + \|G(Y(\cdot, \mu), \cdot + \Delta t, \mu + \Delta\mu) - G(Y(\cdot, \mu), \cdot, \mu)\|] \\
&\leq \alpha L\|Y(\cdot + \Delta t, \mu + \Delta\mu) - Y(\cdot, \mu)\| \\
&\quad + \alpha\|G(Y(\cdot, \mu), \cdot + \Delta t, \mu + \Delta\mu) - G(Y(\cdot, \mu), \cdot, \mu)\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|Y(\cdot + \Delta t, \mu + \Delta\mu) - Y(\cdot, \mu)\| \leq (1 - \alpha L)^{-1}\alpha\|G(&Y(\cdot, \mu), \\
&\cdot + \Delta t, \mu + \Delta\mu) - G(Y(\cdot, \mu), \cdot, \mu)\|.
\end{aligned}$$

It follows from the inequality above that the function Y is uniformly continuous on $\mathbb{R} \times B$.

Note that, for fixed $\mu \in B$,

$$Y(\cdot, \mu) = Y_1(\cdot, \mu) + Y_2(\cdot, \mu),$$

where $Y_1(\cdot, \mu) \in \mathcal{AP}(\mathbb{R})^n$ and $Y_2(\cdot, \mu) \in \mathcal{PAP}_0(\mathbb{R})^n$. Next, we want to show that Y_1 is uniformly continuous on $\mathbb{R} \times B$, and so is Y_2. Since the function

$$Y_1(\cdot + \Delta t, \mu + \Delta\mu) - Y_1(\cdot, \mu) \in \mathcal{AP}(\mathbb{R})^n,$$

it follows from Theorem 1.4 that

$$Y_1(t + \Delta t, \mu + \Delta\mu) - Y_1(t, \mu) \in \overline{\{Y(s + \Delta t, \mu + \Delta\mu) - Y(s, \mu) : s \in \mathbb{R}\}}.$$

Therefore,

$$|Y_1(t + \Delta t, \mu + \Delta\mu) - Y_1(t, \mu)| \leq \sup_{s\in\mathbb{R}} |Y(s + \Delta t, \mu + \Delta\mu) - Y(s, \mu)|.$$

So, the uniform continuity of Y_1 on $\mathbb{R} \times B$ follows from that of Y.

Since both Y_1 and Y_2 are uniformly continuous on $\mathbb{R} \times B$ and B is compact, one sees that $Y_1 \in \mathcal{AP}(\mathbb{R} \times B)^n$ and $Y_2 \in \mathcal{PAP}_0(\mathbb{R} \times B)^n$. The proof is complete.

Let Ω be a ball in $\mathbb{C}^n$ with center at orgin and radius r. The last system we will consider in this section is the form

$$\frac{dY}{dt} = A(t)Y + G \circ (Y \times \iota), \tag{3.6}$$

where $A(t)$ is a $n \times n$ almost periodic matrix and $G \in \mathcal{PAP}(\Omega \times \mathbb{R})^n$. Set

$$B = \{F \in \mathcal{PAP}(\Omega \times \mathbb{R})^n : F(\mathbb{R}) \subset \Omega\}.$$

B is a close subset of $\mathcal{PAP}(\Omega \times \mathbb{R})^n$. Therefore B is a complete metric space.

THEOREM 3.3. *Let Ω, G, $A(t)$, and B be as in the previous paragraph. Assume that* (2.2) *satisfies an exponential dichotomy and the function G satisfies*

$$\left(\frac{k_1}{\sigma_1} + \frac{k_2}{\sigma_2}\right) \sup_{(Z,t)\in\Omega\times\mathbb{R}} |G(Z, t)| \leq r$$

and

$$|G(Z', t) - G(Z'', t)| \leq L|Z' - Z''|, \tag{3.7}$$

with $(k_1/\sigma_1 + k_2/\sigma_2)L < 1$. *Then* (3.6) *has a unique solution in* $\mathcal{PAP}(\mathbb{R})^n$.

Proof. By Theorem 2.3, one can define the mapping $T: B \to \mathcal{PAP}(\mathbb{R})^n$ by the fact that, for $F \in B$, TF is the unique pseudo almost periodic solution of the system

$$\frac{dY}{dt} = A(t)Y + G \circ (F \times \iota). \tag{3.8}$$

We claim that $TB \subset B$. For, by (2.5),

$$\|TF\| \leq \left(\frac{k_1}{\sigma_1} + \frac{k_2}{\sigma_2}\right) \|G \circ (F \times \iota)\| \leq r.$$

The mapping is a contraction on B. In fact, for $F_1, F_2 \in B$, it follows from (2.5) and (3.7) that

$$\begin{aligned} \|TF_1 - TF_2\| &\leq \left(\frac{k_1}{\sigma_1} + \frac{k_2}{\sigma_2}\right) \|G \circ (F_1 \times \iota) - G \circ (F_1 \times \iota)\| \\ &\leq \left(\frac{k_1}{\sigma_1} + \frac{k_2}{\sigma_2}\right) L\|F_1 - F_2\|. \end{aligned}$$

Therefore T has a unique fixed point in B, which is the unique solution of (3.6). The proof is complete.

4. General Ordinary Differential Equations

For $\Omega \in \mathbb{C}^n$, let vector function $F \in \mathcal{PAP}(\Omega \times \mathbb{R})^n$, i.e., $F = G + \Phi$, where $G \in \mathcal{AP}(\Omega \times \mathbb{R})^n$ and $\Phi \in \mathcal{PAP}_0(\Omega \times \mathbb{R})^n$. In this section, we always assume that F is continuous in $Z \in K$ uniformly in $t \in \mathbb{R}$ for all compact subsets $K \subset \Omega$. For the systems

$$\frac{dY}{dt} = F \circ (Y \times \iota) \tag{4.1}$$

$$\frac{dY}{dt} = G \circ (Y \times \iota), \tag{4.2}$$

we will set up some theorems of the existence of pseudo almost periodic solutions of (4.1) or (4.2) implying the existence of almost periodic solutions of (4.2). We start with the following lemma.

LEMMA 4.1. *Suppose that both function f and its derivative f' are in $\mathcal{PAP}(\mathbb{R})$. That is, $f = g + \varphi$ and $f' = \alpha + \beta$, where g, $\alpha \in \mathcal{AP}(\mathbb{R})$ and φ, $\beta \in \mathcal{PAP}_0(\mathbb{R})$. Then the functions g and φ are differentiable so that*

$$g' = \alpha, \qquad \varphi' = \beta.$$

Proof. Note that, for $h \in \mathbb{R}$,

$$\begin{aligned} f_h(t) = f(t+h) - f(t) &= \int_t^{t+h} \alpha(s)\,ds + \int_t^{t+h} \beta(s)\,ds \\ &= I(t) + II(t). \end{aligned}$$

We claim that $I \in \mathcal{AP}(\mathbb{R})$ and $II \in \mathcal{PAP}_0(\mathbb{R})$. We show the assertion only for the case that $h > 0$. Similarly one shows the case of $h < 0$. If τ is an ε-translation of α, then

$$|I(t+\tau) - I(t)| \le \int_t^{t+h} |\alpha(s+\tau) - \alpha(s)|\,ds \le \varepsilon h.$$

Therefore, $I \in \mathcal{AP}(\mathbb{R})$. Now, we show that $II \in \mathcal{PAP}_0(\mathbb{R})$:

$$\begin{aligned} \frac{1}{2T}\int_{-T}^{T} |II(t)|\,dt &\le \frac{1}{2T}\int_{-T}^{T} dt \int_t^{t+h} |\beta(s)|\,ds \\ &= \frac{1}{2T}\left\{\int_{-T}^{-T+h}\int_{-T}^{s} + \int_{-T+h}^{T}\int_{s-h}^{s} \right. \\ &\qquad \left. + \int_T^{T+h}\int_{s-h}^{T} |\beta(s)|\,dt\,ds\right\} \\ &= \frac{1}{2T}\left\{\int_{-T}^{-T+h} |\beta(s|(s+T)\,ds \right. \\ &\qquad \left. + \int_{-T+h}^{T} |\beta(s)|h\,ds + \int_T^{T+h} |\beta(s)|(T-s+h)\,ds\right\} \\ &\le \frac{1}{2T}\left\{\int_{-T}^{-T+h} |\beta(s)|(-T+h+T)\,ds \right. \\ &\qquad \left. + \int_{-T+h}^{T} |\beta(s)|h\,ds + \int_T^{T+h} |\beta(s)|(T-T+h)\,ds\right\} \\ &= \frac{1}{2T}\int_{-T}^{T+h} |\beta(s)|\;|h|\,ds. \end{aligned}$$

It follows that $II \in \mathcal{PAP}_0(\mathbb{R})$ since $\beta \in \mathcal{PAP}_0(\mathbb{R})$.

Note that

$$f_h(t) = [g(t+h) - g(t)] + [\varphi(t+h) - \varphi(t)]$$
$$= g_h(t) + \varphi_h(t).$$

By the uniqueness of the decomposition of a pseudo almost periodic function (Theorem 1.5), we have

$$g_h(t) = g(t+h) - g(t) = \int_t^{t+h} \alpha(s)\,ds$$

and

$$\varphi_h(t) = \varphi(t+h) - \varphi(t) = \int_t^{t+h} \beta(s)\,ds.$$

It follows that $g' = \alpha$ and $\varphi' = \beta$.

THEOREM 4.2. *If the system* (4.1) *has a pseudo almost periodic solution* Y_F, *then the almost periodic component* Y_G *of* Y_F *is a solution of* (4.2).

Proof. Since $Y_F \in \mathcal{PAP}(\mathbb{R})^n$, $Y_F = Y_G + Y_\Phi$, where $Y_G \in \mathcal{AP}(\mathbb{R})^n$ and $Y_\Phi \in \mathcal{PAP}_0(\mathbb{R})^n$. We want to show that Y_G is a solution of (4.2).

By Theorem 1.9, $F \circ (Y_F \times \iota) \in \mathcal{PAP}(\mathbb{R})^n$. By (4.1),

$$\frac{dY_F}{dt} = F \circ (Y_F \times \iota) = G \circ (Y_G \times \iota)$$
$$+ [G \circ (Y_F \times \iota) - G \circ (Y_G \times \iota) + \Phi(Y_F \times \iota)].$$

As in the proof of Theorem 1.9, one sees that $G \circ (Y_G \times \iota) \in \mathcal{AP}(\mathbb{R})^n$ and $[G \circ (Y_F \times \iota) - G \circ (Y_G \times \iota) + \Phi(Y_F \times \iota)] \in \mathcal{PAP}_0(\mathbb{R})^n$. It follows from the lemma that

$$\frac{dY_G}{dt} = G \circ (Y_G \times \iota).$$

The proof is complete.

Fink in [3, Theorem 9.2] and Yoshizawa in [5, Theorem 16.1] state that if (4.2) has an asymptotically almost periodic solution Y, then the almost periodic component of Y is also a solution of (4.2). Since asymptotically almost periodic functions are special cases of pseudo almost periodic functions, the following corollary generalizes this results.

COROLLARY 4.3. *Suppose that the system* (4.2) *has a solution* $Y \in \mathcal{PAP}(\mathbb{R})^n$, *then the almost periodic component of* Y *is also a solution of* (4.2).

Proof. Note that G is uniformly continuous on $K \times \mathbb{R}$ for all compact subsets $K \subset \Omega$. We apply the theorem by putting $\Phi = 0$ in (4.1).

For $G \in \mathcal{AP}(\Omega \times \mathbb{R})^n$, recall that the hull $H(G)$ of G consists of those functions W on $\Omega \times \mathbb{R}$ to which there exists a sequence s_n such that $|G(Z, t + s_n) - W(Z, t)| \to 0$ uniformly on $K \times \mathbb{R}$ as $n \to \infty$, where K is any compact subset of Ω. Now we have the following theorem.

THEOREM 4.4. *If* Y_0 *is a pseudo almost periodic solution of* (4.2) *and* Y_1 *is the almost periodic component of* Y_0. *Then, for every* $W \in H(G)$, *there exists a sequence* $\{s_n\}$ *such that* $Y_1(\cdot + s_n)$ *tends to an almost periodic solution of*

$$\frac{dY}{dt} = W \circ (Y \times \iota)$$

in the norm topology as $n \to \infty$.

Proof. Since $W \in H(G)$, there is a sequence $\{s_n\}$ such that $|G(Z, t + s_n) - W(Z, t)| \to 0$ uniformly in $(Z, t) \in K \times \mathbb{R}$ as $n \to \infty$ for any compact subset K of Ω. Since $Y_1 \in \mathcal{AP}(\mathbb{R})^n$, we assume without loss of generality that $|Y_1(t + s_n) - L(t)| \to 0$ as $n \to \infty$ uniformly for a function $L \in \mathcal{AP}(\mathbb{R})^n$.

By Corollary 4.3, Y_1 is an almost periodic solution of (4.2). So, $Y_1'(t) = G(Y_1(t), t)$ and $Y_1'(t + s_n) = G(Y_1(t + s_n), t + s_n)$. It follows that

$$L'(t) = W(L(t), t).$$

The proof is complete.

Finally, we give an example to show that some solutions are pseudo almost periodic and not asymptotically almost periodic even though the functions are asymptotically almost periodic.

For $n = 1, 2, \ldots$, define

$$\varphi_n(n^3 + in) = \begin{cases} 0, & i = 0, 2, 4, \\ \dfrac{1}{n}, & i = 1, \\ -\dfrac{1}{n}, & i = 3, \end{cases}$$

and φ_n is linear if $t \in [n^3 + in, n^3 + (i+1)n]$ for $i = 0, 1, 2, 3$. Let $f(t) = f(-t)$ if $t < 0$ and for $t > 0$,

$$f(t) = \begin{cases} \varphi_n(t), & t \in [n^3, n^3 + 4n], n = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

Since $f(t) \to 0$ as $|t| \to \infty$, we have that $f \in \mathscr{C}_0(\mathbb{R}) \subset \mathscr{AAP}(\mathbb{R})$. The function f is not in $\mathscr{AP}(\mathbb{R})$ because $\mathscr{AP}(\mathbb{R}) \cap \mathscr{C}_0(\mathbb{R}) = \{0\}$. The solution of the equations

$$y' = f, \qquad y(0) = 0$$

is

$$y(t) = \int_0^t f(s)\,ds \qquad (t \in \mathbb{R}).$$

Then $0 \leq y(t) \leq 1$ for $t \in \mathbb{R}$.

We want to show that $y \in \mathscr{PAP}_0(\mathbb{R})$ and $\notin \mathscr{AAP}(\mathbb{R})$. In fact, for $t > 5$ let k be the largest integer such that $k^3 + 4k \leq t$. Then

$$\begin{aligned} \frac{1}{t}\int_0^t y(s)\,ds &< \frac{1}{t}\sum_{n=1}^{k+1}\int_{n^3}^{n^3+4n} y(s)\,ds \\ &= \frac{1}{t}\sum_{n=1}^{k+1} 2n \\ &< \frac{(k+1)(k+2)}{k^3}. \end{aligned}$$

It follows from the inequalities above that $y \in \mathscr{PAP}_0(\mathbb{R})$.

Now we show that $y \notin \mathscr{AAP}(\mathbb{R})$. If $y \in \mathscr{AAP}(\mathbb{R})$ then $y = g + \varphi$, where $g \in \mathscr{AP}(\mathbb{R})$ and $\varphi \in \mathscr{C}_0(\mathbb{R})$. Since $y \in \mathscr{PAP}_0(\mathbb{R})$ and $\mathscr{AP}(\mathbb{R}) \cap \mathscr{PAP}_0(\mathbb{R}) = \{0\}$ (Theorem 1.5), we have $g = 0$. Thus, $y = \varphi \in \mathscr{C}_0(\mathbb{R})$. But this conflicts the fact that $y(n^3 + 2n) = 1$ for all positive integers n. This contradiction shows that $y \notin \mathscr{AAP}(\mathbb{R})$.

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